

SOME REMARKS ON THE n -LINEAR HILBERT TRANSFORM FOR $n \geq 4$

CAMIL MUSCALU

ABSTRACT. We prove that for every integer $n \geq 4$, the n -linear operator whose symbol is given by a product of two generic symbols of n -linear Hilbert transform type, does not satisfy any L^p estimates of Hölder type. Then, we extend this result to multilinear operators whose symbols are given by a product of an arbitrary number of generic symbols of n -linear Hilbert transform type. These counterexamples are in sharp contrast with the bilinear case, where similar operators are known to satisfy many such L^p estimates.

1. INTRODUCTION

Let $n \geq 2$. For $\vec{\alpha} = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbf{R}^{n-1}$ an arbitrary vector, consider the expression

$$\int_{\mathbf{R}^n} \text{sgn}(\xi + \alpha_1 \xi_1 + \dots + \alpha_{n-1} \xi_{n-1}) \widehat{f}(\xi) \widehat{f_1}(\xi_1) \dots \widehat{f_{n-1}}(\xi_{n-1}) e^{2\pi i x(\xi + \xi_1 + \dots + \xi_{n-1})} d\xi d\xi_1 \dots d\xi_{n-1} \quad (1)$$

where f, f_1, \dots, f_{n-1} are all Schwartz functions on the real line and x is a real number. If all the entries $(\alpha_j)_j$ are different than 0 and 1 and also different from each other, the n -linear operator from (1) is called *the n -linear Hilbert transform* and it will be denoted by $nHT_{\vec{\alpha}}(f, f_1, \dots, f_{n-1})(x)$. Notice that if one erases the symbol

$$\text{sgn}(\xi + \alpha_1 \xi_1 + \dots + \alpha_{n-1} \xi_{n-1}) \quad (2)$$

from (1), the corresponding expression becomes the product of the functions involved

$$f(x) f_1(x) \dots f_{n-1}(x).$$

The main question about these operators is whether they satisfy estimates of Hölder type, more precisely if there exist $1 < p_1, \dots, p_n \leq \infty$ and $0 < p < \infty$ with $1/p_1 + \dots + 1/p_n = 1/p$ so that $nHT_{\vec{\alpha}}$ can be naturally extended as a bounded n -linear operator from $L^{p_1} \times \dots \times L^{p_n}$ into L^p .

The interest in their study comes from their close connection to the so called Calderón commutators [1], [3]. Indeed, a direct calculation shows that modulo a universal constant, one has the identity

$$\int_{[0,1]^{n-1}} nHT_{\vec{\alpha}}(f, a, \dots, a)(x) d\vec{\alpha} = p.v. \int_{\mathbf{R}} \frac{(A(x) - A(y))^{n-1}}{(x - y)^n} f(y) dy \quad (3)$$

with $A' = a$. As one can recognize, the expression on the right hand side is precisely the $(n-1)$ th commutator of Calderón [1], [3].

If $n = 2$ and α is an arbitrary real number different than 0 and 1, *the bilinear Hilbert transform* $2HT_{\alpha}$ does satisfy such estimates, thanks to the work of Lacey and Thiele from [8] and [9]. But for $n \geq 3$ no positive results are presently known.

More generally, for any $k \geq 1$ and arbitrary vectors $\vec{\alpha}_1, \dots, \vec{\alpha}_k \in \mathbb{R}^{n-1}$ denote now by $nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}$ the n -linear operator defined by the *product symbol*

$$\prod_{j=1}^k \text{sgn}(\xi + \alpha_{j1}\xi_1 + \dots + \alpha_{jn-1}\xi_{n-1}) \quad (4)$$

where $\vec{\alpha}_j := (\alpha_{j1}, \dots, \alpha_{jn-1})$ for $j = 1, \dots, k$.

Theorem 1.1. *For any $k \geq 1$ and generic numbers $\alpha_1, \dots, \alpha_k$, the bilinear operator $2HT_{\alpha_1, \dots, \alpha_k}$ satisfies many L^p estimates of Hölder type ¹.*

It is a very simple exercise to show that this theorem follows from the $k = 1$ case studied in [8] and [9] ². See also [6] for some related results.

The goal of this paper is to show that if $k \geq 2$ and n is large enough, the most natural n -linear generalization of the above bilinear theorem, is false. More precisely, we will prove

Theorem 1.2. *For any $k \geq 2$ and generic vectors $\vec{\alpha}_1, \dots, \vec{\alpha}_k$ the n -linear operator $nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}$ does not satisfy any L^p estimates of Hölder type, as long as $n \geq \frac{(2k)!}{(k!)^2} - 1$.*

This time, for $n \geq 3$, the geometry of the symbols (4) becomes more complicated and there do not seem to be any direct connections between the $k = 1$ and $k \geq 2$ cases. In particular, the case $k = 1$ and $n \geq 3$ remains open ³. Another observation that we would like to make is that the condition $n \geq \frac{(2k)!}{(k!)^2} - 1$ is not sharp ⁴. For instance, when $k = 2$, the expression $\frac{(2k)!}{(k!)^2} - 1$ is equal to 5, but we will remark later on that Theorem 1.2 remains valid even for $n = 4$. It is also interesting to compare these negative results with the positive ones in [15], [16] and [17].

The rest of the paper is devoted to the proof of Theorem 1.2. The method we use is a generalization of the arguments from [14] and [18]. See also [5], for some somewhat related ideas.

Acknowledgement: The present work has been partially supported by the NSF.

2. SOME HEURISTICAL ARGUMENTS

Before starting the actual proof, we would like to describe a *heuristic proof* of Theorem 1.2 which will motivate the rigorous argument that will be presented afterwards.

First of all, let us observe that $nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}(f, f_1, \dots, f_{n-1})(x)$ admits the alternative *kernel representation*

$$\frac{(-1)^k}{(i\pi)^k} \cdot p.v. \int_{\mathbb{R}^k} f(x + t_1 + \dots + t_k) \prod_{j=1}^{n-1} f_j(x + \alpha_{j1}t_1 + \dots + \alpha_{jk}t_k) \frac{dt_1}{t_1} \dots \frac{dt_k}{t_k}. \quad (5)$$

¹In the bilinear case, the vectors $\vec{\alpha}_j$ being one dimensional, can be identified with the numbers α_j .

²Indeed, if $n = 2$, one can first observe that the symbol (4) is constant on various *angular regions* centered at the origin, and so it is enough to understand bilinear operators whose symbols are the characteristic functions of such *angular sets*. But (modulo some natural compositions with certain Riesz projections) the study of these can be easily reduced to the study of bilinear operators of type $2HT_\alpha$. The details are left to the reader.

³However, in this particular case, it has been noticed (first heuristically in [10] and then rigorously in [4]) that the trilinear Hilbert transform cannot map $L^{p_1} \times L^{p_2} \times L^{p_3}$ into L^p for every $1/3 < p < \infty$. See also [2] for another interesting *tri-linear counterexample*.

⁴As we will see, the method of proof has the potential to give an even lower bound, but we didn't find any value in checking this so carefully, for the moment.

This is a simple consequence of the well known identity $i\pi \operatorname{sgn}(\xi) = \widehat{\frac{1}{t}}(\xi)$ applied k times to the symbol (4). Consider now $f(x) = e^{i\#x^k}$ and $f_j(x) = e^{i\#_j x^k}$ for $j = 1, \dots, n-1$ where $\#, \#_1, \dots, \#_{n-1}$ are real numbers that will be determined later on.

If one plugs in these functions into the formula (5), one formally obtains

$$p.v. \int_{\mathbf{R}^k} e^{i(\#(x+t_1+\dots+t_k)^k + \sum_{j=1}^{n-1} \#_j(x+\alpha_{1j}t_1+\dots+\alpha_{kj}t_k)^k)} \frac{dt_1}{t_1} \dots \frac{dt_k}{t_k}.$$

The new expression

$$\#(x+t_1+\dots+t_k)^k + \sum_{j=1}^{n-1} \#_j(x+\alpha_{1j}t_1+\dots+\alpha_{kj}t_k)^k \quad (6)$$

should be interpreted as a polynomial in the $k+1$ variables x, t_1, \dots, t_k which is homogeneous of degree k . An elementary combinatorial computation shows that this expression has precisely $\frac{(2k)!}{(k!)^2}$ monomials. For reasons that will be clearer a bit later, we would like to choose our numbers $\#, \#_1, \dots, \#_{n-1}$ in such a way that all the coefficients of these monomials are zero with the exception of the ones corresponding to x^k and $t_1 \cdot \dots \cdot t_k$. Let us have a look at the coefficient of t_1^k for instance. It is given by

$$\# + \sum_{j=1}^{n-1} \#_j \alpha_{1j}^k$$

and so the fact that it is zero is equivalent to the fact that the n -dimensional vector $(\#, \#_1, \dots, \#_{n-1})$ is orthogonal to $(1, \alpha_{11}^k, \dots, \alpha_{1n-1}^k)$. Since one can argue in a similar way for all the other monomials, our wish becomes equivalent to the fact that $(\#, \#_1, \dots, \#_{n-1})$ is orthogonal to $\frac{(2k)!}{(k!)^2} - 2$ other vectors in \mathbf{R}^n . Since $\vec{\alpha}_1, \dots, \vec{\alpha}_k$ are generic, all these vectors will be linearly independent and the fact that such a vector exists is guaranteed by the condition $n \geq \frac{(2k)!}{(k!)^2} - 1$ stated in Theorem 1.2. Furthermore, by a proper dilation, one can also assume that the coefficient of $t_1 \cdot \dots \cdot t_k$ will be equal to 1.

In particular, for any such a vector $(\#, \#_1, \dots, \#_{n-1})$ one can write

$$|nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}(f, f_1, \dots, f_{n-1})(x)| = \frac{1}{\pi^k} \cdot \left| \int_{\mathbf{R}^k} \frac{e^{it_1 \cdot \dots \cdot t_k}}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \right|. \quad (7)$$

On the other hand, the right hand side of (7) can be further calculated as

$$\left| \int_{\mathbf{R}^k} \frac{e^{it_1 \cdot \dots \cdot t_k}}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \right| = \pi \int_{\mathbf{R}^{k-1}} \frac{\operatorname{sgn}(t_1 \cdot \dots \cdot t_{k-1})}{t_1 \cdot \dots \cdot t_{k-1}} dt_1 \dots dt_{k-1} = \pi \left(\int_{\mathbf{R}} \frac{1}{|t|} dt \right)^{k-1}$$

and this means that formally, we have obtained the identity

$$|nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}(f, f_1, \dots, f_{n-1})(x)| = \frac{1}{\pi^{k-1}} \cdot |f(x)f_1(x)\dots f_{n-1}(x)| \cdot \left(\int_{\mathbf{R}} \frac{1}{|t|} dt \right)^{k-1}. \quad (8)$$

Notice that while the moduli of the initial functions are all equal to 1, the right hand side of (8) is infinite. The idea now is to restrict all the functions above to an interval of type $[-N, N]$ and to observe that as long as x belongs to an interval of the same size, one has

$$|nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}(f, f_1, \dots, f_{n-1})(x)| \geq c(\log N)^{k-1} \quad (9)$$

as $N \rightarrow \infty$. Clearly, (9) would imply Theorem 1.2 and from now on the goal is to describe a rigorous proof it ⁵.

3. PROOF OF THEOREM 1.2

Fix $\vec{\alpha}_1, \dots, \vec{\alpha}_k$ generic vectors in \mathbf{R}^{n-1} . For N large enough, define the function $\chi_N(x)$ to be the characteristic function of the interval $[-N, N]$ and $\widetilde{\chi}_N(x)$ to be a smooth function supported on $[-N - \epsilon, N + \epsilon]$ and equal to 1 on $[-N, N]$, where $\epsilon > 0$ is a number much smaller than $1/N^{k-1}$.

Consider real numbers $\#, \#_1, \dots, \#_{n-1}$ chosen to satisfy all the requirements of Section 2. Define the functions f, f_1, \dots, f_{n-1} by

$$f(x) = \widetilde{\chi}_N(x)e^{i\#x^k}$$

and

$$f_j(x) = \widetilde{\chi}_N(x)e^{i\#_j x^k}$$

for $1 \leq j \leq n-1$. We claim that there exist small constants c and \widetilde{c} depending on all these parameters with the exception of N so that

$$|nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}(f, f_1, \dots, f_{n-1})(x)| \geq c(\log N)^{k-1} \quad (10)$$

as long as $x \in [-\widetilde{c}N, \widetilde{c}N]$. Clearly, as we pointed out earlier, (10) would immediately imply Theorem 1.2, since it holds for arbitrarily large N .

To prove the claim let us first observe that since f, f_1, \dots, f_{n-1} are smooth and compactly supported, formula (5) can be applied and one has

$$\begin{aligned} nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}(f, f_1, \dots, f_{n-1})(x) = \\ e^{i(\# + \#_1 + \dots + \#_{n-1})x^k} \frac{(-1)^k}{(i\pi)^k} \int_{\mathbf{R}^k} \widetilde{\chi}_N(x + t_1 + \dots + t_k) \prod_{j=1}^{n-1} \widetilde{\chi}_N(x + \alpha_{1j}t_1 + \dots + \alpha_{kj}t_k) \frac{e^{it_1 \dots t_k}}{t_1 \dots t_k} dt_1 \dots dt_k. \end{aligned} \quad (11)$$

By splitting $e^{it_1 \dots t_k}$ as

$$e^{it_1 \dots t_k} = \cos(t_1 \dots t_k) + i \sin(t_1 \dots t_k)$$

and ignoring the harmless factor $e^{i(\# + \#_1 + \dots + \#_{n-1})x^k}$, one can decompose the rest of (11) as

$$\frac{(-1)^k}{(i\pi)^k} \int_{\mathbf{R}^k} \widetilde{\chi}_N(x + t_1 + \dots + t_k) \prod_{j=1}^{n-1} \widetilde{\chi}_N(x + \alpha_{1j}t_1 + \dots + \alpha_{kj}t_k) \frac{\cos(t_1 \dots t_k)}{t_1 \dots t_k} dt_1 \dots dt_k \quad (12)$$

+

⁵The functions f, f_1, \dots, f_{n-1} which appeared in (9) are the old ones restricted smoothly to an interval of type $[-N, N]$.

$$i \frac{(-1)^k}{(i\pi)^k} \int_{\mathbf{R}^k} \widetilde{\chi}_N(x + t_1 + \dots + t_k) \prod_{j=1}^{n-1} \widetilde{\chi}_N(x + \alpha_{1j} t_1 + \dots + \alpha_{kj} t_k) \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k. \quad (13)$$

It is a good moment now to pause and make a few important remarks regarding the supports of our integrands in (12) and (13). Consider a generic set of the form

$$\{(t_1, \dots, t_k) \in \mathbf{R}^k : a \leq \beta_1 t_1 + \dots + \beta_k t_k \leq b\} \quad (14)$$

with $a < 0 < b$ and $\vec{\beta} = (\beta_1, \dots, \beta_k)$ arbitrary. Clearly, this set is a k dimensional strip, containing the origin and lying between the hyperspaces $\beta_1 t_1 + \dots + \beta_k t_k = a$ and $\beta_1 t_1 + \dots + \beta_k t_k = b$ which are both perpendicular to the given vector $\vec{\beta}$. Moreover, the *width* of this strip is $O(b - a)$.

As a consequence, the support of the integrands in (12) and (13) lies at the intersection of n such k dimensional strips. Since the vectors $\vec{\alpha}_1, \dots, \vec{\alpha}_k$ are *generic*, if one picks \widetilde{c} small enough and $x \in [-\widetilde{c}N, \widetilde{c}N]$ this intersection will be a bounded domain in \mathbf{R}^k containing the origin and also contained in a large cube of sidelength $O(N)$. Hence, the term (13) is well defined and this means that the term (12) is well defined as well (being the difference of two well defined expressions).

In particular, from (11), (12) and (13) one can see that

$$\begin{aligned} & |nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}(f, f_1, \dots, f_{n-1})(x)| \\ & \geq \frac{1}{\pi^k} \left| \int_{\mathbf{R}^k} \widetilde{\chi}_N(x + t_1 + \dots + t_k) \prod_{j=1}^{n-1} \widetilde{\chi}_N(x + \alpha_{1j} t_1 + \dots + \alpha_{kj} t_k) \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \right| \end{aligned} \quad (15)$$

for every $x \in [-\widetilde{c}N, \widetilde{c}N]$.

Next, we would like to observe that modulo some harmless *error terms*, one can replace all the $\widetilde{\chi}_N$ functions in (15) by the corresponding χ_N . To see this, let us denote the n linear inner expression in (15) by $\mathcal{E}(\widetilde{\chi}_N, \widetilde{\chi}_N, \dots, \widetilde{\chi}_N)(x)$. One can write

$$\begin{aligned} & \mathcal{E}(\widetilde{\chi}_N, \widetilde{\chi}_N, \dots, \widetilde{\chi}_N)(x) \\ & = \mathcal{E}(\chi_N, \widetilde{\chi}_N, \dots, \widetilde{\chi}_N)(x) + \mathcal{E}(\widetilde{\chi}_N - \chi_N, \widetilde{\chi}_N, \dots, \widetilde{\chi}_N)(x) \end{aligned}$$

and it is not difficult to see that the absolute value of the error term $\mathcal{E}(\widetilde{\chi}_N - \chi_N, \widetilde{\chi}_N, \dots, \widetilde{\chi}_N)(x)$ is at most $O(1)$, as a consequence of the fact that the function $\frac{\sin x}{x}$ is bounded and that $\widetilde{\chi}_N - \chi_N$ is supported on a union of two strips of width $O(1/N^{k-1})$. Iterating this argument n times one obtains from (15) that

$$\begin{aligned} & |nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}(f, f_1, \dots, f_{n-1})(x)| \\ & \geq \frac{1}{\pi^k} \left| \int_{\mathbf{R}^k} \chi_N(x + t_1 + \dots + t_k) \prod_{j=1}^{n-1} \chi_N(x + \alpha_{1j} t_1 + \dots + \alpha_{kj} t_k) \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \right| - O(1) \end{aligned} \quad (16)$$

for every $x \in [-\widetilde{c}N, \widetilde{c}N]$.

Clearly, the inner term on the right hand side of (16) can be written as

$$\int_{D_x} \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \quad (17)$$

where D_x is a compact and convex domain in \mathbf{R}^k containing the origin and having also the property that

$$[-c_1 N, c_1 N]^k \subseteq D_x \subseteq [-C_1 N, C_1 N]^k$$

where c_1 is small and C_1 is large and they depend on $\vec{\alpha}_1, \dots, \vec{\alpha}_k$ but are otherwise independent on $x \in [-\widetilde{c}N, \widetilde{c}N]$.

Split now (17) as

$$\begin{aligned} & \int_{D_x} \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \\ &= \int_{[-c_1 N, c_1 N]^k} \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k + \int_{D_x \setminus [-c_1 N, c_1 N]^k} \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k. \end{aligned} \quad (18)$$

We will prove in the next two sections that the first term in (18) is positive and is *bounded from below* by $c(\log N)^{k-1}$ while the absolute value of the second term in (18) is *bounded from above* by $C(\log N)^{k-2}$.

Combining these two facts with the previous (16) will imply the desired (10).

4. LOWER LOGARITHMIC BOUNDS

In this section we prove the lower logarithmical bounds for the first term in (18) that have been mentioned at the end of the previous Section 3.

Proposition 4.1. *For any integer $k \geq 1$ there exists a constant $c(= c(k))$ with the property that*

$$\int_{-N}^N \dots \int_{-N}^N \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \geq c(\log N)^{k-1} \quad (19)$$

as long as N is large enough.

The proof of this Proposition 4.1 is based on the following lemma whose enuntiation requires some additional notations.

If f is a bounded measurable function defined on the interval $[0, \infty)$ we denote by $Hf(x)$ the linear operator given by

$$Hf(x) = \frac{1}{x} \int_0^x f(u) du$$

for every $x \in [0, \infty)$. We will also denote by H^l the composition of H with itself l times (as long as l is an integer greater or equal than 1) and by H^0 the identity operator.

Lemma 4.2. *For any integer $k \geq 1$ there exist a small constant c_{k-1} and a large one C_{k-1} with the property that*

$$\int_0^t H^{k-1} F(u) du \geq c_{k-1} (\log t)^{k-1} \quad (20)$$

for every $t \geq C_{k-1}$, where $F(u) := \frac{\sin u}{u}$.

Let us first assume this Lemma 4.2 and show how our previous Proposition 4.1 can be reduced to it.

If a is any real number different than zero, the function $s \rightarrow \frac{\sin as}{s}$ is an even function and in particular this implies that

$$\int_{-N}^N \frac{\sin as}{s} ds = 2 \int_0^N \frac{\sin as}{s} ds.$$

Using this observation several times, one can see that

$$\int_{-N}^N \dots \int_{-N}^N \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k = 2^k \int_0^N \dots \int_0^N \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k. \quad (21)$$

We claim now that the following identity holds

$$\int_0^N \dots \int_0^N \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k = \int_0^{N^k} H^{k-1} F(u) du. \quad (22)$$

Clearly, if we take this equality for granted then (22), (21) and (20) together imply the desired (19).

For simplicity, we will prove (22) in the particular case $k = 3$ and leave the general case to the reader, since it does not require any additional ideas.

One can write

$$\begin{aligned} \int_0^N \int_0^N \int_0^N \frac{\sin(t_1 t_2 t_3)}{t_1 t_2 t_3} dt_1 dt_2 dt_3 &= \int_0^N \int_0^N \left(\int_0^N \frac{\sin(t_1 t_2 t_3)}{t_3} dt_3 \right) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \\ &= \int_0^N \int_0^N \left(\int_0^{N t_1 t_2} \frac{\sin x}{x} dx \right) \frac{dt_2}{t_2} \frac{dt_1}{t_1} = \int_0^N \int_0^{N^2 t_1} \left(\int_0^y \frac{\sin x}{x} dx \right) \frac{dy}{y} \frac{dt_1}{t_1} \\ &= \int_0^{N^3} \int_0^z \left(\int_0^y \frac{\sin x}{x} dx \right) \frac{dy}{y} \frac{dz}{z} = \int_0^{N^3} \left(\frac{1}{z} \int_0^z \left(\frac{1}{y} \int_0^y \frac{\sin x}{x} dx \right) dy \right) dz \\ &= \int_0^{N^3} H^2 F(z) dz \end{aligned}$$

as desired.

We are left with the proof of Lemma 4.2. We proceed by induction. Clearly, the $k = 1$ case is a simple consequence of the fact that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$. Suppose now that (20) holds for the parameter $k - 1$ and we would like to prove it for k .

One writes

$$\int_0^t H^k F(u) du = \int_0^t \left(\frac{1}{x} \int_0^x H^{k-1} F(u) du \right) dx \quad (23)$$

$$= \int_0^{C_{k-1}} \dots + \int_{C_{k-1}}^t \dots \quad (24)$$

The absolute value of the first term in (24) is clearly at most C_{k-1} given that $|F(u)| \leq 1$. Using the induction hypothesis on the other hand, one can estimate the second term in (24) from below by

$$\begin{aligned} c_{k-1} \int_{C_{k-1}}^t \frac{1}{x} (\log x)^{k-1} dx &= \frac{c_{k-1}}{k} \int_{C_{k-1}}^t [(\log x)^k]' dx \\ &= \frac{c_{k-1}}{k} ((\log t)^k - (\log C_{k-1})^k). \end{aligned}$$

All of these imply that the left hand side of (23) can be estimated from below by

$$= \frac{c_{k-1}}{k} ((\log t)^k - (\log C_{k-1})^k) - C_{k-1}.$$

But this expression is at least as big as $\frac{c_{k-1}}{2k} (\log t)^k$ if t is large enough and this completes the proof of Lemma 4.2 and therefore of Proposition 4.1.

5. UPPER LOGARITHMIC BOUNDS

Our final goal now is to prove the upper logarithmical bounds for the second term in (18), that have been claimed in Section 3.

Proposition 5.1. *Let D be a compact and convex domain in \mathbf{R}^k having the property that*

$$[-cN, cN]^k \subseteq D \subseteq [-CN, CN]^k$$

where c and C are fixed constants and N is large enough. Then, there exists $\widetilde{C}(= \widetilde{C}(k))$ so that

$$\left| \int_{D \setminus [-cN, cN]^k} \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \right| \leq \widetilde{C} (\log N)^{k-2}. \quad (25)$$

We claim that the above Proposition 5.1 follows easily from the following

Lemma 5.2. *Let D be a compact and convex domain in \mathbf{R}^k having the property that*

$$D \subseteq [-N, N]^k.$$

Then, there exists $C(= C(k))$ so that

$$\left| \int_D \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \right| \leq C (\log N)^{k-1}. \quad (26)$$

Let us see first why Lemma 5.2 implies Proposition 5.1.

If $\vec{t} = (t_1, \dots, t_k) \in D \setminus [-cN, cN]^k$ then at least for one index $1 \leq i \leq k$ one must have $t_i \notin [-cN, cN]$. But then, this means that $t_i \in [-CN, -cN] \cup [cN, CN]$.

Let us examine now the following two *extremal* cases.

Suppose first that $t_i \in [-CN, -cN] \cup [cN, CN]$ for every $1 \leq i \leq k$. In this case it is not difficult to see that the integral over that corresponding region is at most

$$\int_{cN}^{CN} \dots \int_{cN}^{CN} \frac{dt_1}{t_1} \dots \frac{dt_k}{t_k}$$

which is clearly bounded by a constant independent of N .

Assume now that we are in the opposite situation when precisely one index i has the property that $t_i \in [-cN, -cN] \cup [cN, CN]$. By symmetry, we can also assume that that index is 1 and that $t_1 \in [cN, CN]$. In this case, it is also not difficult to see that the integral over the corresponding region can be expressed as

$$\int_{cN}^{CN} \frac{1}{t_1} \left(\int_{D_{t_1}} \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_2 \cdot \dots \cdot t_k} dt_2 \dots dt_k \right) dt_1 \quad (27)$$

where

$$D_{t_1} := \{(t_2, \dots, t_k) : (t_1, t_2, \dots, t_k) \in D\}.$$

It is natural to change variables $t_1^{1/k-1} t_j = s_j$ for $2 \leq j \leq k$ and rewrite (27) as

$$\int_{cN}^{CN} \frac{1}{t_1} \left(\int_{\tilde{D}_{t_1}} \frac{\sin(s_2 \cdot \dots \cdot s_k)}{s_2 \cdot \dots \cdot s_k} ds_2 \dots ds_k \right) dt_1 \quad (28)$$

where \tilde{D}_{t_1} is also compact and convex and has the property that

$$\tilde{D}_{t_1} \subseteq [-CNt_1^{1/k-1}, CNt_1^{1/k-1}]^{k-1}.$$

Using Lemma 5.2 one can then bound (28) easily by $C(\log N)^{k-2}$ which is of course acceptable by (25).

The general case when an arbitrary number of indices i satisfy $t_i \in [-cN, -cN] \cup [cN, CN]$ can be treated similarly and the corresponding upper bound will be of the form $C(\log N)^l$ for some $0 \leq l \leq k-2$. Since there are only a finite number of such situations, this completes the proof of (25).

We are left with the proof of Lemma 5.2. We proceed as before by induction.

The case $k = 1$ is obviously true, since D is now an interval $[a, b]$ and (26) becomes equivalent to

$$\left| \int_a^b \frac{\sin x}{x} dx \right| \leq C.$$

Let us consider now the general case of (26) assuming (by the induction hypothesis) that all the previous ones are known.

Decompose the inner integral in (26) as

$$\begin{aligned} & \int_D \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \\ &= \int_{D \cap \{\vec{t} : |\vec{t}|_\infty \leq 1\}} \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k + \sum_{d=0}^{\log N} \int_{D \cap \{\vec{t} : 2^d < |\vec{t}|_\infty \leq 2^{d+1}\}} \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k^6. \end{aligned} \quad (29)$$

⁶We use the notation $|\vec{t}|_\infty := \max_{1 \leq i \leq k} |t_i|$.

Arguing as before and using the induction hypothesis one can see that

$$\left| \int_{D \cap \{i: 2^d < |i|_\infty \leq 2^{d+1}\}} \frac{\sin(t_1 \cdot \dots \cdot t_k)}{t_1 \cdot \dots \cdot t_k} dt_1 \dots dt_k \right| \leq C d^{k-2}. \quad (30)$$

Finally, using (30) in (29) one obtains the desired (26).

6. FURTHER REMARKS

First of all, as we promised, we would like to explain why Theorem 1.2 holds true even for $k = 2$ and $n = 4$. Recall from (5) the kernel representation of $4HT_{\vec{\alpha}_1, \vec{\alpha}_2}(f, f_1, f_2, f_3)(x)$ as

$$-\frac{1}{\pi^2} p.v. \int_{\mathbf{R}^2} \prod_{j=1}^3 f_j(x + \alpha_{1j}s + \alpha_{2j}t) \frac{dt}{t} \frac{ds}{s} \quad (31)$$

where $\vec{\alpha}_1 = (\alpha_{11}, \alpha_{12}, \alpha_{13})$ and $\vec{\alpha}_2 = (\alpha_{21}, \alpha_{22}, \alpha_{23})$ are two generic vectors in \mathbf{R}^3 .

Consider as before $f(x) = e^{i\#x^2}$ and $f_j(x) = e^{i\#_j x^2}$ for $j = 1, 2, 3$ where $\#, \#_1, \#_2, \#_3$ are real numbers that will be determined later on. If one formally plugs in these functions into (31), the corresponding expression in (6) becomes a polynomial in the variables x, t, s which is homogeneous of degree 2. This polynomial has precisely six monomials, namely xt, xs, ts, t^2, s^2 and x^2 each of which having its corresponding coefficient. We would like to choose our numbers $\#, \#_1, \#_2, \#_3$ so that the coefficients of xt, xs and s^2 are all zero. As we discussed earlier in Section 2, this amounts to pick a vector $(\#, \#_1, \#_2, \#_3) \in \mathbf{R}^4$ orthogonal to three other generic linearly independent 4 dimensional vectors, which is clearly possible. Using this choice, the analogous of (7) becomes

$$|4HT_{\vec{\alpha}_1, \vec{\alpha}_2}(f, f_1, f_2, f_3)(x)| = \frac{1}{\pi^2} \left| \int_{\mathbf{R}^2} e^{i\alpha t^2} e^{i\beta ts} \frac{dt}{t} \frac{ds}{s} \right| \quad (32)$$

where α, β are real numbers depending on the previous parameters $\vec{\alpha}_1, \vec{\alpha}_2$ and $\#, \#_1, \#_2, \#_3$. By construction, one can also assume without loss of generality that $\alpha > 0$. As in Section 2 one then observes that the expression on the right hand side of (32) can be calculated further as

$$\begin{aligned} \left| \int_{\mathbf{R}^2} e^{i\alpha t^2} e^{i\beta ts} \frac{dt}{t} \frac{ds}{s} \right| &= \frac{1}{\pi} \left| \int_{\mathbf{R}} \frac{\text{sgn}(t)}{t} e^{i\alpha t^2} dt \right| \\ &= \frac{1}{\pi} \left| \int_0^\infty \frac{e^{i\alpha t}}{t} dt \right| = \frac{1}{\pi} \left| \int_0^\infty \frac{\cos t}{t} dt + i \frac{\pi}{2} \right| \end{aligned}$$

and while $\int_1^\infty \frac{\cos t}{t} dt$ is bounded, the integral $\int_0^1 \frac{\cos t}{t} dt$ is infinite.

To transform this heuristical argument into a rigorous one, one proceeds as before. The details are left to the reader.

Let us end with a few remarks on the previous identity (3). First of all, it can be suggestively rewritten as

$$\int_{[0,1]^{n-1}} nHT_{\vec{a}}(f, a, \dots, a)(x) d\vec{a} = p.v. \int_{\mathbf{R}} \left(\frac{\Delta_t}{t} A(x) \right)^{n-1} f(x+t) \frac{dt}{t} \quad (33)$$

where in general $\Delta_t g(x) := g(x+t) - g(x)$ is the finite difference of the function g at scale t . In [12] the following generalization of it has been noticed

$$\begin{aligned} & \int_{[0,1]^{n-1}} \cdots \int_{[0,1]^{n-1}} nHT_{\vec{\alpha}_1, \dots, \vec{\alpha}_k}(f, a, \dots, a)(x) d\vec{\alpha}_1 \dots d\vec{\alpha}_k \\ &= p.v. \int_{\mathbf{R}^k} \left(\frac{\Delta_{t_1}}{t_1} \circ \dots \circ \frac{\Delta_{t_k}}{t_k} A(x) \right)^{n-1} f(x+t_1+\dots+t_k) \frac{dt_1}{t_1} \dots \frac{dt_k}{t_k} \end{aligned} \quad (34)$$

where this time $A^{(k)} = a$. It is interesting to mention that the linear operators on the right hand side of (34) are bounded on L^p for every $1 < p < \infty$ as long as $A^{(k)} \in L^\infty$.

These operators appeared naturally in [12] as part of a generalization of Calderón's theory to classes of functions having arbitrary *polynomial growth*. For more details, the reader is referred to the recent sequel of the author [11], [12] and [13].

REFERENCES

- [1] Calderón, A., *Commutators, singular integrals on Lipschitz curves and applications*, Proc. Int. Congress of Math., Helsinki, 1978, Academia Scientiarum Fennica, Helsinki, 85-96, [1980].
- [2] Christ, M., *On certain elementary trilinear operators*, Math. Res. Lett., vol. 8, 43-56, [2001].
- [3] Coifman, R. and Meyer, Y., *Wavelets, Calderón-Zygmund operators and multilinear operators*, Cambridge Studies in Advanced Mathematics, xv+314 pp., [1997].
- [4] Demeter, C., *Divergence of combinatorial averages and the unboundedness of the trilinear Hilbert transform*, Ergodic Theory Dynam. Systems, 1453-1464, [2008].
- [5] Fefferman, C., *On the divergence of multiple Fourier series*, Bull. Amer. Math. Soc., vol. 77, 191-195, [1971].
- [6] Gilbert, J. and Nahmod, A., *Bilinear operators with non-smooth symbols*, J. Fourier Anal. Appl., vol. 7, 435-467, [2001].
- [7] Kenig, C. and Stein, E., *Multilinear operators and fractional integration*, Math. Res. Lett., vol. 6, 1-15, [1999].
- [8] Lacey M., Thiele C., *L^p estimates for the bilinear Hilbert transform for $2 < p < \infty$* , Ann. of Math., vol. 146, 693-724, [1997].
- [9] Lacey M., Thiele C., *On Calderón's conjecture*, Ann. of Math., vol. 149, 475-496, [1999].
- [10] Muscalu, C., *Unpublished notes*, IAS Princeton, [2003].
- [11] Muscalu, C., *Calderón commutators and the Cauchy integral on Lipschitz curves revisited I. First commutator and generalizations*, ArXiv:1201.3845, 23 pages, [2012].
- [12] Muscalu, C., *Calderón commutators and the Cauchy integral on Lipschitz curves revisited II. Cauchy integral and generalizations*, ArXiv:1201.3850, 29 pages, [2012].
- [13] Muscalu, C., *Calderón commutators and the Cauchy integral on Lipschitz curves revisited III. Polydisc extensions*, ArXiv:1201.3855, 25 pages, [2012].
- [14] Muscalu, C., Tao, T. and Thiele, C., *A counterexample to a multilinear endpoint question of Christ and Kiselev*, Math. Res. Lett. 10, 237-246, [2003].
- [15] Muscalu, C., Tao, T. and Thiele, C., *L^p estimates for the biest I. The Walsh case* Math. Ann., vol. 329, 401-426, [2004].
- [16] Muscalu, C., Tao, T. and Thiele, C., *L^p estimates for the biest II. The Fourier case* Math. Ann., vol. 329, 427-461, [2004].
- [17] Muscalu, C., Tao, T. and Thiele, C., *Multilinear multipliers associated to simplexes of arbitrary length* ArXiv:0712.2420v1, [2007].
- [18] Muscalu C., Pipher J., Tao T., Thiele C., *Bi-parameter paraproducts*, Acta Math., vol. 193, 269-296, [2004].